Partial Differentiation

Dr. G.V.S.R. Deekshitulu
Professor of Mathematics
JNT University Kakinada
Kakinada, INDIA

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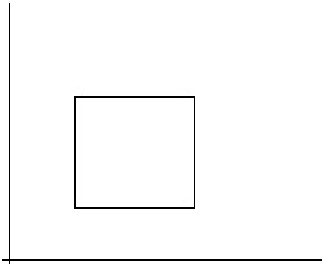
Till now we have focussed on functions with one independent variable. If $u = f(x_1, x_2, ..., x_n)$ where $x_1, x_2, ..., x_n$ are independent variables, then u is called a multi variable function with n variables. Anyhow, we restrict our discussion mostly to two and three variable functions.

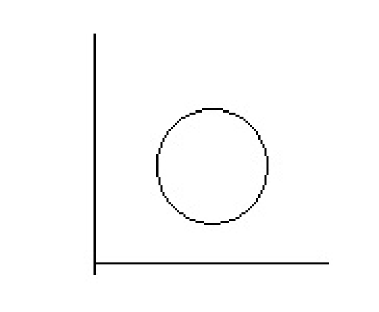
δ -neighborhood of a point in a plane

δ- neighbourhood of a point (a, b) in xy-plane is a square bounded by $x = a - \delta, x = a + \delta, y = b - \delta$ and$y = a + \delta$ i.e.,
$a - \delta \leq x \leq a + \delta, b - \delta \leq y \leq b + \delta.$



Open disc

Neighbourhood of a point (a, b) may also be defined as an open circular disc with centre at (a, b) and radius δ. i.e., $(x - a)^2 + (y - b)^2 < \delta^2$.



Limits

$u = f(x, y)$ is said to have limit L as (x, y) approaches (a, b) and is denoted by $\underset{(x,y)\to(a,b)}{lim} f(x, y) = L$
if, for given $\epsilon > 0$, we can find a δ such that $|f(x) - L| < \epsilon, \forall x, y$ in the δ -neighbourhood $|x - a| < \delta$ and $|y - b| < \delta$ $[or (x - a)^2 + (y - b)^2 < \delta^2.]$

The limit of the function is said to exist only when the limit along along path in xy-plane from (x, y) to (a, b) is same. Otherwise the limit does not exist.

Properties:

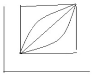
As (x, y) tends to (a, b), if $lim f(x, y) = L$ and $lim g(x, y) = M$, then as (x, y) tends to (a, b)

i. $lim[f(x) \pm g(x)] = L \pm M$

ii. $lim[f(x.g(x)] = L.M$

iii. $lim[\frac{f(x)}{g(x)}] = \frac{L}{M}, M \neq 0$

Procedure:



Evaluate

i. limit $f(x, y)$ as $x \to a$ and $y \to b$

ii. limit $f(x, y)$ as $y \to b$ and $x \to a$

iii. if $a = b = 0$, limit $f(x, y)$ along $y = mx$ or $y = mx^n$

If the limit in all the above cases is same, then the limit exists.

Example 1 If $f(x,y) = \frac{y^2-x^2}{x^2+y^2}$, find *lim* as (x, y) tends to $(0, 0)$

Solution: i). As $x \to 0$, $lim f(x,y) = \frac{y^2}{y^2} = 1$.

ii). As $y \to 0$, $lim f(x,y) = \frac{-x^2}{x^2} = -1$.

The limit is not the same in the above cases. Therefore limit does not exist.

Example 2. Evaluate $\lim\limits_{\substack{x\to1\\y\to2}} \dfrac{2x^2y}{x^2+y^2+1}$.

Solution : $\lim\limits_{\substack{x\to1\\y\to2}} \dfrac{2x^2y}{x^2+y^2+1} = \lim\limits_{x\to1}\left\{\lim\limits_{y\to2}\left(\dfrac{2x^2y}{x^2+y^2+1}\right)\right\} = \lim\limits_{x\to1}\dfrac{4x^2}{x^2+5} = \dfrac{4}{6} = \dfrac{2}{3}$

or $\lim\limits_{\substack{x\to1\\y\to2}} \dfrac{2x^2y}{x^2+y^2+1} = \lim\limits_{y\to2}\left\{\lim\limits_{x\to1}\dfrac{2x^2y}{x^2+y^2+1}\right\} = \lim\limits_{y\to2}\dfrac{2y}{y^2+2} = \dfrac{4}{6} = \dfrac{2}{3}$

Example 3 : If $f(x,y) = \dfrac{x-y}{2x+y}$ show that $\lim\limits_{x\to0}\left\{\lim\limits_{y\to0} f(x,y)\right\} \neq \lim\limits_{y\to0}\left\{\lim\limits_{x\to0} f(x,y)\right\}$

Solution : $\lim\limits_{x\to0}\left\{\lim\limits_{y\to0} f(x,y)\right\} = \lim\limits_{x\to0}\left\{\lim\limits_{y\to0}\dfrac{x-y}{2x+y}\right\} = \lim\limits_{x\to0}\dfrac{x}{2x} = \dfrac{1}{2}$ (cancelling x)

$\lim\limits_{y\to0}\left\{\lim\limits_{x\to0} f(x,y)\right\} = \lim\limits_{y\to0}\left\{\lim\limits_{x\to0}\dfrac{x-y}{2x+y}\right\} = \lim\limits_{y\to0}\dfrac{-y}{y} = -1$ (cancelling y).

Hence the result follows.

Continuity

A function $f(x,y)$ is said to be continuous at (a,b) if $limf(x,y) = f(a,b)$ as $x \to a$ and $y \to b$.

Ex.1 Discuss the continuity of$f$ defined as $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 2$.

Solution: i). As $x \to 0$, $limf(x,y) = 0$
ii). As $y \to 0$, $limf(x,y) = \frac{x}{x} = 1$

The limit is not the same in the above cases. Therefore limit does not exist. Hence not continuous at $(0,0)$.

Example : Examine for continuity at the origin of the function defined by

$$f(x, y) = \frac{x^2}{\sqrt{x^2+y^2}} \quad \text{for } x \neq 0, y \neq 0$$

$$\text{for } x = 0, y = 0.$$

Redefine the function to make it continuous.

Solution : Notice that the value of $f(x, y)$ for $x = 0, y = 0$ is not given in the problem. Let us discuss the continuity of the given function at (0, 0).

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x^2}{\sqrt{x^2+y^2}}\right\} = \lim_{x\to 0}\left\{\frac{x^2}{x}\right\} = \lim_{x\to 0} x = 0$$

Also, $$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+y^2}}\right\} = \lim_{y\to 0}\left\{\frac{0}{\sqrt{0+y^2}}\right\} = \lim_{y\to 0}(0) = 0$$

$$\therefore \lim_{x\to 0}\left\{\lim_{y\to 0}(f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}.$$

Also along the path $y = mx$,

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\frac{x^2}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0}\frac{x}{\sqrt{1+m^2}} = 0$$

Similarly, along the path $y = mx^2$,

$$\lim_{x\to 0} f(x,y) = 0$$

Hence the function $f(x, y)$ is continuous at the origin if $f(x, y) = 0$ for $x = 0, y = 0$. Otherwise $f(x, y)$ is not continuous at the origin.

If $f(x, y)$ is not continuous at (0, 0) then define $f(x, y) = 0$ for $x = 0, y = 0$ so that $f(x, y)$ is continuous at origin.

Example 3 : Discuss the continuity of the function

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Solution : Let us consider the limit of the function for testing the continuity along the line $y=mx$.

$$\text{Now } \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{2xy}{x^2+y^2}=\lim_{x\to 0}\frac{2mx^2}{x^2+m^2x^2}=\frac{2m}{1+m^2}$$

which is different for the different m selected.

$\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

Consider

$$\lim_{x\to 0} f(x,0)=\lim_{x\to 0}\frac{2x(0)}{x^2+0}=\lim_{x\to 0} 0=0=f(0,0)$$

$$\lim_{y\to 0} f(0,y)=\lim_{y\to 0}\frac{2.0y}{0+y^2}=\lim_{y\to 0} 0=0=f(0,0)$$

$\therefore f(x,y)$ is continuous for given values of x and y but it is not continuous at $(0,0)$.

Ex.4:
Given that $f(x, y) = \begin{cases} x^3 + 3y^2 + 2x + y, & \text{if } (x, y) \neq (2, 3) \\ 10, & \text{if } (x, y) = (2, 3) \end{cases}$

Find the limit of $f(x, y)$ at (2, 3)

$$\lim_{\substack{x\to 2 \\ y\to 3}} f(x, y) = \lim_{\substack{x\to 2 \\ y\to 3}} (x^3 + 3y^2 + 2x + y)$$

$$= \lim_{x\to 2}\{\lim_{y\to 3}(x^3 + 3y^2 + 2x + y)\} = \lim_{x\to 2}(x^3 + 2x + 30) = 42$$

$$\lim_{\substack{x\to 2 \\ y\to 3}} f(x, y) = \lim_{\substack{x\to 2 \\ y\to 3}} (x^3 + 3y^2 + 2x + y)$$

$$= \lim_{y\to 3}\{\lim_{x\to 2}(x^3 + 3y^2 + 2x + y)\} = \lim_{y\to 3}(3y^2 + y + 12) = 42$$

Hence from equation (1) and (2) limit exist and equal to 42

$$\lim_{\substack{x\to 2 \\ y\to 3}} f(x, y) = 42 \neq f(2, 3)$$

Therefore the function is discontinuous at (2, 3).

Let $u = f(x, y, z)$. If we Keep y *and* z as constants and vary x, then the derivative of u with respect to x is called **Partial derivative** of u with respect to x[denoted by $\frac{\partial u}{\partial x}$] and is defined as

$$\frac{\partial u}{\partial x} =_{h\to 0}^{lim} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

Similarly partial derivatives with respect to y and z are also defined and are denoted as $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ respectively.

Higher order partial derivatives

The higher order partial derivatives of $u = f(x, y, z)$ are obtained by successive differentiation. i.e.,

$\frac{\partial^2 u}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}[\frac{\partial u}{\partial x}]$, $\frac{\partial^2 u}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}[\frac{\partial u}{\partial y}]$, $\frac{\partial^2 u}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}[\frac{\partial u}{\partial y}]$, and $\frac{\partial^2 u}{\partial y \partial x} = f_{xy} = \frac{\partial}{\partial y}[\frac{\partial u}{\partial x}]$.

Mixed partial derivatives $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ are equal if the derivatives involved are continuous.

Total derivative

Let $u = f(x, y, z)$, then the **Total derivative** of f, denoted by df, is defined as

$$df = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

Chain Rule

Let $u = f(x, y, z)$ and x, y, z are again functions of t. Then f can be treated as a function of t. Then the derivative of u w.r.t t is called **Total derivative w.r.t.** t and is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Ex. 1. Find the total derivative of $u = x^2 - y^2$ where $x = e^t cost, y = e^t sint$ at $t = 0$.

Solution: We know that $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.

$$\begin{aligned} \frac{du}{dt} &= 2x.e^t[cost - sint] - 2ye^t[sint + cost] \\ \frac{du}{dt}|_{t=0} &= [2e^{2t}cost[cost - sint] - 2e^{2t}sint[sint + cost]]_{t=0} = 2. \end{aligned}$$

Ex. 2. Find $\frac{du}{dt}$ of $u = ln(x + y + z)$, where $x = e^{-t}, y = sint, z = cost$

Solution: $\frac{du}{dt} = \frac{1}{x+y+z}[-e^{t} + cost - sint]$.

Ex. Find $\frac{du}{dt}$ for the following function: $u = \sin\left(\frac{x}{y}\right), x = e^{t}, y = t^2$

Solution:

Given that $u = \sin\left(\frac{x}{y}\right), x = e^{t}, y = t^2$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial x}{\partial y}\frac{dy}{dt}$$

$$= \cos\left(\frac{x}{y}\right)\cdot\frac{1}{y}\cdot e^{t} + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)\cdot(2t)$$

$$= \cos\left(\frac{e^{t}}{t^2}\right)\left[\frac{e^{t}}{t^2} - \frac{e^{t}}{t^4}\cdot 2t\right] = \cos\left(\frac{e^{t}}{t^2}\right)\frac{e^{t}}{t^3}(t-2)$$

Ex. If $u = x\log(xy)$, where $x^3 + y^3 + 3xy = 1$, then find $\frac{du}{dx}$

Solution:

Given that $x^3 + y^3 + 3xy = 1$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

From $x^3 + y^3 + 3xy - 1 = 0 = f(x, y)$

$$fx = 3x^2 + 3y, fy = 3y^2 + 3x$$

$$\frac{dy}{dx} = \frac{-fx}{fy} = \frac{-(x^2 + y)}{(y^2 + x)}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{\partial u}{\partial x} = (x)\log(xy) + \frac{x}{xy}\cdot y = 1 + \log(xy), \quad \frac{\partial u}{\partial y} = \frac{(x)}{xy}(x) = \frac{1}{y}$$

$$\therefore \quad \frac{du}{dx} = 1 + \log(xy) + \frac{x}{y}\cdot\frac{-(x^2 + y)}{(y^2 + x)} = 1 + \log(xy) - \frac{x(x^2 + y)}{y(y^2 + x)}$$

Example 6 : If $x^x y^y z^z = e$ show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Solution : Given that $x^x y^y z^z = e$

Taking logarithm on both sides, we get $x \log x + y \log y + z \log z = \log e = 1$

$\Rightarrow \quad z \log z = 1 - x \log x - y \log y$

Differentiating partially w.r.t. 'x', we get

$$\left(z.\frac{1}{z} + 1.\log z\right)\frac{\partial z}{\partial x} = -\left(x.\frac{1}{x} + 1.\log x\right)$$

$$\Rightarrow (1 + \log z)\frac{\partial z}{\partial x} = -(1 + \log x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)} \qquad \text{...(1)}$$

Similarly, $\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{1 + \log z}$ ...(2)

When $x = y = z$, we have

$$\frac{\partial z}{\partial x} = -1 \text{ and } \frac{\partial z}{\partial y} = -1$$

Now differentiating (2) partially w.r.t. 'x', we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)}\right]$$

$$= -(1+\log y)\left[-(1+\log z)^{-2}\frac{1}{z}\frac{\partial z}{\partial x}\right] = \frac{1+\log y}{z(1+\log z)^2}\frac{\partial z}{\partial x} \quad \text{...(3)}$$

When $x = y = z$ from (3), we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1+\log x}{x(1+\log x)^2}(-1) \left(\because \frac{\partial z}{\partial x} = -1\right)$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e + \log x)} \quad (\because \quad \log e = 1)$$

$$= -\frac{1}{x \log ex} = -(x \log ex)^{-1}$$

Ex. If $x^x y^y z^z = c$, then show that $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$ at $(x = y = z)$

Solution:

Given that $x^x y^y z^z = c$

$$x \log x + y \log y + z \log z = \log c$$

Differentiating equation (1) with respect to y we get

$$y\frac{1}{y} + \log y(1) + z \cdot \frac{1}{z}\frac{\partial z}{\partial y} + \log z\frac{\partial z}{\partial y} = 0$$

$$(1 + \log y) + (1 + \log z)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-[1 + \log y]}{1 + \log z}$$

Differentiating equation (1) with respect to x we get $\frac{\partial z}{\partial x} = \frac{-[1 + \log x]}{1 + \log z}$

Now Differentiating equation (2) with respect to 'x' we get

$$(1 + \log z)\frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial y}\right)\left[\frac{1}{z} \cdot \frac{\partial z}{\partial y}\right] = 0$$

$$(1 + \log z)\frac{\partial^2 z}{\partial xy} + \frac{1}{z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 0$$

at $x = y = z, \dfrac{\partial z}{\partial y} = -1, \dfrac{\partial z}{\partial x} = -1$

$$(1 + \log z)\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{z}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{z[1 + \log z]} = \frac{-1}{x[1 + \log x]}$$

$$= \frac{-1}{x[\log_e^e + \log x]} = \frac{-1}{x \log ex} = -[x \log ex]^{-1}$$

Differentiation of implicit functions

An implicit function of $x\&y$ is of the form $f(x,y)=0$ which can not be solve for one variable in terms of the other explicitly. Then differentiating w.r.t x, we get $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$.

Which implies $\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$.

Ex 1. Find $\frac{dy}{dx}$ if $f(x,y)=xsin(x-y)-(x+y)$.

Solution: $\frac{\partial f}{\partial x}=sin(x-y)+xcos(x-y)-1$ and $\frac{\partial f}{\partial y}=-xcos(x-y)-1$.

Therefore $\frac{dy}{dx}=\frac{sin(x-y)+xcos(x-y)-1}{xcos(x-y)+1}$.

Ex.2.Find $\frac{dy}{dx}$ if $y^{xy} = sinx$.

Solution: Taking ln on both sides $x^y lny = ln(sinx)$.

Let $z = x^y$, then $lnz = ylnx$. Then $\frac{1}{z}z_x = \frac{y}{x}$ implies that
$z_x = \frac{yz}{x} = \frac{y}{x}x^y = yx^{y-1}$.
And $\frac{1}{z}z_y = lnx$. Implies that $z_y = zlnx = x^y lnx$
Now ,consider $f(x, y) = x^y lny - ln(sinx)$.
$\frac{\partial f}{\partial x} = \frac{\partial x^y}{\partial x} lny - \frac{cosx}{sinx} = yx^{y-1} lny - \frac{cosx}{sinx}$ and
$\frac{\partial f}{\partial y} = \frac{\partial x^y}{\partial y} lny + \frac{x^y}{y} = x^y lnxlny + \frac{x^y}{y}$.
Now $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{yx^{y-1} lny - cotx}{x^y lnxlny + \frac{x^y}{y}}$.

Homogeneous function:

A polynomial in x and y is said to be homogeneous if all the terms are of same degree. i.e.,

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

(Or) A function $f(x, y)$ is said to be homogeneous of degree n if it can be expressed as $f(x, y) = x^n \phi(\frac{y}{x})$ or $y^n \phi(\frac{x}{y})$

Example:

$3x^2 - 2xy + 15y^2 = x^2(3 - 2(\frac{y}{x}) + 15(\frac{y}{x})^2) = x^2\phi(\frac{y}{x})$ That means $f(x, y)$ is homogeneous function in x and y with degree 2.

Ex.

3. Let $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$. Now $\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = z$ (say).

$$\therefore \quad z = \frac{x^{1/2}\left[1-\left(\frac{y}{x}\right)^{1/2}\right]}{x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]} = x^0 g\left(\frac{y}{x}\right).$$

$\therefore$ z *i.e.* $\sin u$ is a homogeneous function of order 0.

Euler's theorem:

If f is a homogeneous function of x, y of degree n then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

Example:

If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ find the value of $xu_x + yu_y$.

Solution:

$\frac{x^2+y^2}{x+y} = e^u = f$, which is a homogeneous function of degree 1.
By Euler's theorem, $xu_x + yu_y = nu$
$f = e^u \Rightarrow f_x = e^u u_x, f_y = e^u u_y$.
$\therefore xf_x + yf_y = nf \Rightarrow xu_x e^u + yu_y e^u = f = e^u$
$xu_x + yu_y = 1$.

Example 2:

Find the value of $xu_x + yu_y + zu_z$ if $u = cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$.

Solution:

Let $f = cosu = \frac{x^3+y^3+z^3}{ax+by+cz} = x^2\phi(\frac{y}{x}, \frac{z}{x})$.
$\therefore$ homogeneous function of degree 2.

$$\therefore xf_x + yf_y + zf_z = 2f.$$
$$f_x = \frac{\partial f}{\partial u}.u_x = -sinuu_x$$
$$f_y = -sinuu_y, f_z = -sinuu_z$$
$$\therefore -[xu_x + yu_y + zu_z]sinu = 2cosu$$
$$\Rightarrow xu_x + yu_y + zu_z = -2cotu.$$

Example:

Find the value of $xu_x + yu_y + zu_z$ if $u = \frac{y}{z} + \frac{z}{x}$.

Ex.3 Given that $u(x, y, z) = \frac{y}{z} + \frac{z}{x}$

$$u(kx, ky, kz) = \frac{ky}{kz} + \frac{kz}{kx} = k^0 u$$

Hence u is a homogeneous function of degree $(n) = 0$

By Euler's theorem we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Theorem : If $u(x, y)$ is a homogeneous function of degree 'n' in x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Ex.
Find $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$ using Euler's theorem for the following function

$u = x^2\tan(y/x) - y^2\tan^{-1}(x/y)$

Sol.

Given that $u = x^2\tan(y/x) - y^2\tan^{-1}(x/y)$

$$u(kx, ky) = k^2(x^2\tan(y/x) - y^2\tan^{-1}(x/y))$$

Hence u is a Homogeneous function degree 2.

By Euler's theorem

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

Differentiating (1) with respect to x we get

$$x\frac{\partial^2 u}{\partial y^2}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}$$

Differentiating (1) with respect to y we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial y}$$

Multiply equation (2) with x and (3) with y and add we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

Ex. Find $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$ using Euler's theorem for the following function

$$u=\log\left(\frac{x^2+y^2}{x+y}\right)$$

Sol.

Given that $u=\log\left(\frac{x^2+y^2}{x+y}\right)\Rightarrow e^u=\frac{x^2+y^2}{x+y}$

$$e^{u(kx,ky)}=k\left(\frac{x^2+y^2}{x+y}\right)=ke^u$$

e^u is a Homogenous function of degree 1.

By Euler's theorem we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

$$xe^u\frac{\partial u}{\partial x}+y\cdot e^u\frac{\partial u}{\partial y}=e^u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$$

$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)(1)=0$$

Jacobian:

Let $u(x, y, z)$ and $v(x, y, z)$ be functions of two independent variables x and y. The Jacobian of u and v w.r.to x, y and z is denoted by $J(\frac{u,v,w}{x,y,z})$ (or) $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ is defined as

$$J(\frac{u, v, w}{x, y, z}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Properties:

(i.) If $J = \frac{\partial(u,v)}{\partial(x,y)}$, $J^* = \frac{\partial(x,y)}{\partial(u,v)}$, then $JJ^* = 1$.

(ii.) If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \frac{\partial(r,s)}{\partial(x,y)}$$

(iii.) If $x = rcos\theta, y = rsin\theta$ (polar coordinates) then $\frac{\partial(x,y)}{\partial(r,\theta)} = r$.

(iv.) If $x = rcos\theta, y = rsin\theta, z = z$ (Cylindrical coordinates) then $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$.

(v.) If $x = rsin\theta cos\phi, y = rsin\theta sin\phi, z = rcos\theta$ (Spherical coordinates) then $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 sin\theta$.

Example 1:

$x = e^u cosv, y = e^u sinv$ find $\frac{\partial(x,y)}{\partial(u,v)}$.

Solution:

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

where

$$x_u = e^u cosv, x_v = -e^u sinv, y_u = e^u sinv, y_v = e^u cosv.$$

$$\therefore J = \begin{vmatrix} e^u cosv & -e^u sinv \\ e^u sinv & e^u cosv \end{vmatrix} = e^{2u}.$$

Ex. Evaluate the following using the relation $JJ^1 = 1$

If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$, then find $J\left(\frac{x, y, z}{u, v, w}\right)$

Solution:

Given that $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$

Since $JJ^1 = 1$, then $J^1 = 1/J$

Now $$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right] = 0 + 2 + 2 = 4$$

Hence $J^1 = 1/J = 1/4$

i.e., $$J\left(\frac{x, y, z}{u, v, w}\right) = \frac{1}{4}$$

Ex. Evaluate the following using the second property of jacobian:

If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$, then find $J\left(\frac{u, v}{r, \theta}\right)$

Solution:

Given that $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$

Then $J\left(\frac{u, v}{r, \theta}\right) = J\left(\frac{u, v}{x, y}\right) \cdot J\left(\frac{x, y}{r, \theta}\right)$

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4(x^2 + y^2) = 4r^2$$

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Hence $J\left(\frac{u, v}{r, \theta}\right) = J\left(\frac{u, v}{x, y}\right) \cdot J\left(\frac{x, y}{r, \theta}\right) = -4r^2 \cdot r = -4r^3$

Functional dependence:

The functions $u = f(x, y)$ and $v = g(x, y)$ are said to be functionally dependent on one another if $J = \frac{\partial(u,v)}{\partial(x,y)} = 0$
If $J \neq 0$, they are functionally independent.

Example :

$$u = e^x siny, v = e^x cosy$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} e^x siny & e^x cosy \\ e^x cosy & -e^x siny \end{vmatrix} = -e^{2x}[sin^2 y + cos^2 y] \neq 0$$

$\therefore$ functionally independent.

Example:

Whether $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ and $v = \sin^{-1}(x) + \sin^{-1}(y)$ are functionally dependent or not, if so find the relation between them.

Given that $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1}(x) + \sin^{-1}(y)$

If u and v are functionally dependent, then $J\left(\frac{u, v}{x, y}\right) = 0$

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= 1 - \frac{xy}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}} - 1 + \frac{xy}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}} = 0$$

Hence u and v are functionally dependent and the relation is given by

$$v = \sin^{-1}(x) + \sin^{-1}(y) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \sin^{-1}(u)$$

$\Rightarrow$ $u = \sin v$

Example:

Whether $u = x^2e^{-2y}\cos hz$, $v = x^2e^{-y}\sin hz$ and $w = 3x^4e^{-2y}$ are functionally dependent or not, if so find the relation between them

Sol. Given that $u = x^2e^{-y}\cosh z$, $v = x^2e^{-y}\sinh z$ and $w = 3x^4e^{-2y}$

If u, v and w are functionally dependent, then $J\left(\frac{u, v, w}{x, y, z}\right) = 0$

$$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2xe^{-y}\cosh z & -x^2e^{-y}\cosh z & x^2e^{-y}\sinh z \\ 2xe^{-y}\sinh z & -x^2e^{-y}\sinh z & x^2e^{-y}\cosh z \\ 12x^3e^{-2y} & -6x^4e^{-2y} & 0 \end{vmatrix}$$

$$= 2xe^{-y}\cosh z(6x^6e^{-3y}\cosh z) + x^2e^{-y}\cosh z(-12x^5e^{-3y}\cosh z)$$
$$+ x^2e^{-y}\sinh z(-12x^5e^{-3y}\sin h\, z + 12x^5e^{-3y}\cosh z) = 0$$

Since $J\left(\frac{u, v, w}{x, y, z}\right) = 0$, u, v and w are functionally dependent and the relation is given by $3u^2 - 3v^2 = w$

Taylor's Theorem for function of two variable:

Let $f(x, y)$ be a function of two variables, then the Taylor's series expansion about $x = a, y = b$ is

$$f(x+h, y+k) = f(a, b)+hf_x(a, b)+kf_y(a, b)+\frac{1}{2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a, b)+...$$

Maclaurin's Theorem for function of two variable:

It is a special case of Taylor's series, when the expansion is about $a = 0, b = 0. \therefore \quad f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}\left[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right] + ...$

Example : Expand $f(x,y) = e^y \log(1+x)$ in powers of x and y using Maclaurin's Series

Solution : We are given

$$f(x,y) = e^y \log(1+x)\,; \quad f(0,0) = 1 \qquad f_{yy}(x,y) = e^y \log(1+x)\,; \quad f_{yy}(0,0) = 0$$

$$\therefore \quad f_x(x,y) = e^y \cdot \frac{1}{1+x}\,; \quad f_x(0,0) = 1 \qquad f_{xxy}(x,y) = -\frac{e^y}{(1+x)^2}\,; \quad f_{xxy}(0,0) = -1$$

$$f_y(x,y) = e^y \log(1+x)\,; \quad f_y(0,0) = 0 \qquad f_{xyy}(x,y) = \frac{e^y}{1+x}\,; \quad f_{xyy}(0,0) = 1$$

$$f_{xy}(x,y) = \frac{e^y}{1+x}\,; \quad f_{xy}(0,0) = 1 \qquad f_{xxx}(x,y) = \frac{2e^y}{(1+x)^3}\,; \quad f_{xxx}(0,0) = 2$$

$$f_{xx}(x,y) = -\frac{e^y}{(1+x)^2}\,; \quad f_{xx}(0,0) = -1 \qquad f_{yyy}(x,y) = e^y \log(1+x)\,; \quad f_{yyy}(0,0) = 0$$

$\therefore$ By Maclaurin's series,

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \ldots$$

Expand e^{xy} in powers of $(x-1)$ and $(y-1)$.

Sol. Given that $f(x, y) = e^{xy}$, $f(1, 1) = e$

By Taylor's series about (x_0, y_0) we have

$$\begin{aligned} f(x, y) = {} & f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \\ & + \frac{1}{2!}((x - x_0)^2 f_{xx}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \\ & + 2(x - x_0)(y - y_0) f_{xy}(x_0, y_0)) + \dots \end{aligned} \tag{1}$$

Here $x_0 = 1$, $y_0 = 1$

Calculate the following

$$f_x(x, y) = ye^{xy}, \; f_x(1, 1) = e$$

$$f_y(x, y) = xe^{xy}, \; f_y(1, 1) = e$$

$$f_{xx}(x, y) = y^2 e^{xy}, \; f_{xx}(1, 1) = e$$

$$f_{yy}(x, y) = x^2 e^{xy}, \; f_{yy}(1, 1) = e$$

$$f_{xy}(x, y) = xye^{xy} + e^{xy}, \; f_{xy}(1,1) = 2e$$

substitute the above in equation (1), we get

$$\therefore \; f(x, y) = e + (x-1)e + (y-1)e + \frac{1}{2!}((x-1)^2 e + (y-1)^2 e + 2(x-1)(y-1)2e) + \dots$$

$$= e\left[1 + (x-1) + (y-1) + \frac{1}{2!}((x-1)^2 - (y-1)^2 - 2(x-1)(y-1)) + \dots\right]$$

Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$.

Sol. Given that $f(x, y) = x^2y + 3y - 2$, $f(1, -2) = -10$

By Taylor's series about (x_0, y_0) we have

$$f(x, y) = f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)$$
$$+ \frac{1}{2!}((x - x_0)^2 f_{xx}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0)$$
$$+ 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0)) + \ldots$$

Here $x_0 = 1$, $y_0 = -2$

$$f_x(x, y) = 2xy,\ f_x(1, -2) = -4$$
$$f_y(x, y) = x^2 + 3,\ f_y(1, -2) = 4$$
$$f_{xx}(x, y) = 2y,\ f_{xx}(1, -2) = -4$$
$$f_{yy}(x, y) = 0,\ f_{yy}(1, -2) = 0$$
$$f_{xy}(x, y) = 2x,\ f_{xy}(1, -2) = 2$$

substitute the above in equation (1), we get

$$f(x, y) = -10 + (x - 1)(-4) + (y + 2)(4) + \frac{1}{2!}((x - 1)^2(-4) + (y + 2)^2(0)$$
$$+ 2(x - 1)(y + 2)(2))$$
$$= -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2)$$
$$= 2 - 4x + 4y - 2(x - 1)^2 + 2(x - 1)(y + 2)$$

Maxima and Minima of function of two variables:

Definition : Let $f(x, y)$ be a function of two variables x and y.

At $x = a; y = b, f(x,y)$ is said to have maximum or minimum value, if $f(a, b) > f(a + h, b + k)$ or $f(a, b) < f(a + h, b + k)$ respectively where h and k are small values.

Extreme value : $f(a, b)$ is said to be an extreme value of f, if it is a maximum or minimum value.

(I) The necessary conditions for $f(x, y)$ to have a maximum or minimum at (a, b) are

$$f_x(a, b) = 0; f_y(a, b) = 0$$

(II) Sufficient conditions : Suppose that $f_x(a, b) = 0; f_y(a, b) = 0$ and let

$$\frac{\partial^2}{\partial x^2} f(a,b) = l; \frac{\partial^2}{\partial x \partial y} f(a,b) = m; \frac{\partial^2}{\partial y^2} f(a,b) = n.$$

Then (*i*) $f(a, b)$ is a maximum value if $ln - m^2 > 0$ and $l < 0$.

(*ii*) $f(a, b)$ is a minimum value if $ln - m^2 > 0$ and $l > 0$.

(*iii*) $f(a, b)$ is not an extreme value if $ln - m^2 < 0$.

(*iv*) If $ln - m^2 = 0$, then $f(x, y)$ fails to have maximum or minimum value and it needs further investigation.

Note : Stationary value. $f(a, b)$ is said to be a stationary value of $f(x, y)$ if $f_x(a, b) = 0$; $f_y(a, b) = 0$. Thus every extreme value is a stationary value but the converse may not be true.

Working Rule to find the Maximum or Minimum values of $f(x, y)$:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero. Solve these equations for x and y. Let $(a_1, b_1), (a_2, b_2), \ldots$ be the pairs of values.

2. Find $l = \frac{\partial^2 f}{\partial x^2}$, $m = \frac{\partial^2 f}{\partial x \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$, for each pair of values obtained in step (1).

3. (*i*) If $l\,n - m^2 > 0$ and $l < 0$ at (a_1, b_1), then (a_1, b_1) is a point of maximum and $f(a_1, b_1)$ is maximum value.

   (*ii*) If $l\,n - m^2 > 0$ and $l > 0$ at (a_1, b_1), then (a_1, b_1) is a point of minimum and $f(a_1, b_1)$ is a minimum value.

   (*iii*) If $l\,n - m^2 < 0$ at (a_1, b_1), then $f(a_1, b_1)$ is not an extreme value, *i.e.*, there is neither a maximum nor a minimum at (a_1, b_1). In this case (a_1, b_1) is a saddle point.

   (*iv*) If $l\,n - m^2 = 0$ at (a_1, b_1), no conclusion can be drawn about maximum or minimum and needs further investigation.

   Similarly, examine the other pairs of values $(a_2, b_2), (a_3, b_3), \ldots$ one by one.

Example : Find the maximum and minimum values of $x^3+3xy^2-15x^2-15y^2+72x$

Solution : Let $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$.

Then $\frac{\partial f}{\partial x}=3x^2+3y^2-30x+72$, $\frac{\partial f}{\partial y}=6xy-30y=6y\,(x-5)$

Now $l=\frac{\partial^2 f}{\partial x^2}=6x-30=6\,(x-5)$, $m=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}(6xy-30y)=6y$

and $n=\frac{\partial^2 f}{\partial y^2}=6x-30=6(x-5)$

The critical points of f are given by $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0$

i.e., $3x^2+3y^2-30x+72=0$ and $6y\,(x-5)=0$

i.e., $x^2+y^2-10x+24=0$ and $(y=0$ or $x=5)$

$\Rightarrow (y=0$ and $x^2+y^2-10x+24=0)$ or $(x=5$ and $x^2+y^2-10x+24=0)$

$\Rightarrow (y=0, x^2-10x+24=0)$ or $(x=5, 25+y^2-50+24=0)$

$\Rightarrow (y=0$ and $x=6,4)$ or $(x=5$ and $y=\pm 1)$

$\therefore$ The critical points of f are $A(4,0), B(6,0), C(5,1)$ and $D(5,-1)$

Now, $\delta = ln - m^2 = 36[(x-5)^2 - y^2]$

At $A(4,0), \delta = 36[(4-5)^2 - 0] = 36 > 0$

At $B(6,0), \delta = 36[(6-5)^2 - 0] = 36 > 0$

At $C(5,1), \delta = 36(0-1) = -36 < 0$

At $D(5,-1), \delta = 36(0-1) = -36 < 0$

Thus A and B are points of extremum for f, while C and D are saddle points.

But $l = 6(x-5) = 6(4-5) = -6 < 0$ at $A(4,0)$

$\Rightarrow$ A is the point of maximum for f

and $l = 6(x-5) = 6(6-5) = 6 > 0$ at $B(6,0)$

$\Rightarrow$ B is the point of minimum for f

Minimum value of $f = 4^3 + 3(4)(0) - 15(4)^2 - 15(0) + 72(4) = 112$

Maximum value of $f = 6^3 + 3(6)(0) - 15(6)^2 - 15(0) + 72(6) = 108$

Example : Find the maximum and minimum values of $xy + \frac{a^3}{x} + \frac{a^3}{y}$.

Solution : Given function is $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ ... (1)

$\therefore \quad \frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}, \quad \frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$ and

$\frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$ and $\frac{\partial^2 f}{\partial x \partial y} = 1.$

The condition for $f(x, y)$ to have min. (or) max. is $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$\Rightarrow \quad y = \frac{a^3}{x^2}$ ...(2) and $x = \frac{a^3}{y^2}$ ...(3)

Substituting (3) in (2), we get

$$y = \frac{a^3 y^4}{a^6} = \frac{y^4}{a^3}$$

$$\Rightarrow \quad y(y^3 - a^3) = 0$$

$\Rightarrow \quad y = 0$ or $y = a$

Now $y = 0 \Rightarrow x = \infty$

$\therefore$ It is not possible.

Now $\quad y = a \Rightarrow x = a \quad \therefore$ The extremum point is (a, a)

$f(x, y)$ will have max. or min at (a, a).

At (a, a), $l = \dfrac{\partial^2 f}{\partial x^2} = 2$, $m = 1$, $n = 2$

Now $\; ln - m^2 = 4 - 1 = 3 > 0$, $l = 2 > 0$

$\therefore f(x, y)$ has minimum at (a, a).

The minimum value is $f(a, a) = a^2 + \dfrac{a^3}{a} + \dfrac{a^4}{a} = 3a^2$

Example : Investigate the maxima and minima, if any, of the function $f(x) = x^3y^2(1-x-y)$.

Solution : We have

$$f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2(3-4x-3y)$$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = x^3y(2-2x-3y)$$

$$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1-2x-y).$$

$$m = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}(2x^3y - 2x^4y - 3x^3y^2)$$

$$= 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6-8x-9y)$$

$$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y = 2x^3(1-x-3y)$$

$$\therefore \quad ln - m^2 = 6xy^2(1-2x-y)\cdot 2x^3(1-x-3y) - (x^2y)^2(6-8x-9y)^2$$

$$= (x^2y)^2\left[12(1-2x-y)(1-x-3y) - (6-8x-9y)^2\right]$$

For maxima and minima, $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

$\Rightarrow x^2y^2(3-4x-3y)=0$ and $x^3y(2-2x-3y)=0$

$\Rightarrow x=0, y=0$ or $3-4x-3y=0$ and $x=0, y=0$ or $2-2x-3y=0$

The possible extremum of $f(x,y)$ are

$(x=0, y=0)$, ($x=0$ and $3-4x-3y=0$), ($x=0$ and $2-2x-3y=0$)

($y=0$ and $2-2x-3y=0$) and ($3-4x-3y=0$ and $2-2x-3y=0$)

i.e., $(0,0),(0,1),\left(0,\frac{2}{3}\right),(1,0),(0,1)$ and $\left(\frac{1}{2},\frac{1}{3}\right)$.

At all these points except $\left(\frac{1}{2},\frac{1}{3}\right)$, $ln-m^2=0$ i.e., there is no extremum value

At $\left(\frac{1}{2},\frac{1}{3}\right)$, $ln-m^2=\frac{1}{9.64}>0$ and $l=6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2\left(1-1-\frac{1}{3}\right)=-\frac{1}{9}<0$

$\therefore \left(\frac{1}{2},\frac{1}{3}\right)$ is a point of maximum.

Maximum value $=f\left(\frac{1}{2},\frac{1}{3}\right)=\left(\frac{1}{8}\cdot\frac{1}{9}\right)\left(1-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{72}\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{432}$

Example : Find three positive numbers whose sum is 100 and whose product is maximum.

Solution : Let x, y, z be the required three numbers.

Then $x+y+z=k(=100)$ ... (1)

and $f(x,y,z)=xyz$ ... (2)

Eliminating z from (2) with the help of (1), we get

$$f(x,y)=xy\,(k-x-y)$$

$$\therefore \quad \frac{\partial f}{\partial x}=y[x(-1)+(k-x-y)\cdot 1]=y\,(k-2x-y)$$

$$\frac{\partial f}{\partial y}=x[y(-1)+(k-x-y)\cdot 1]=x\,(k-x-2y)$$

For $f(x,y)$ to be maximum, $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

$\Rightarrow \quad 2x+y=k$ and $x+2y=k$

Solving these, we get $x=y=\frac{k}{3}$

Now $r(\text{or } l)=\frac{\partial^2 f}{\partial x^2}=-2y$, $s(\text{or } m)=\frac{\partial^2 f}{\partial x \partial y}=x(-1)+(k-x-2y)\cdot 1=k-2x-2y$

and $t(\text{or } n)=\frac{\partial^2 f}{\partial y^2}=-2x$

Now $rt - s^2$ (or $ln - m^2$) $= 4xy - (k - 2x - 2y)^2$

At $x = y = \frac{k}{3}$, $rt - s^2 = \frac{4k^2}{9} - \left(k - \frac{2k}{3} - \frac{2k}{3}\right)^2 = \frac{4k^2}{9} - \frac{k^2}{9} = \frac{3k^2}{9} = \frac{k^2}{3} > 0$

Also at $x = y = \frac{k}{3}$, $r = -2y = \frac{-2k}{3} < 0$

Hence $f(x, y)$ has a maximum at $\left(\frac{k}{3}, \frac{k}{3}\right)$.

$\therefore$ From (1), $z = k - (x + y) = k - \frac{2k}{3} = \frac{k}{3}$

The required numbers are $\frac{k}{3}, \frac{k}{3}, \frac{k}{3}$ i.e., $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$ $(\because k = 100)$.

Thus the product is maximum when all the three numbers are equal.

Example : A rectangular box open at the top is to have volume of 32 cubic *ft*. Find the dimensions of the box requiring least material for its construction.

Solution : Let x ft, y ft and z ft be the dimensions of the box and let S be the surface of the box. Then we have

$S = xy + 2yz + 2zx$ (Since open at the top) ...(1)

Given that its volume, $x.y.z = 32$ ...(2)

From (2), $z = \dfrac{32}{xy}$

Substituting the value of z in (1), we get

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x = xy + \frac{64}{x} + \frac{64}{y}$$

Now $\dfrac{\partial S}{\partial x} = y - \dfrac{64}{x^2} = 0$ and $\dfrac{\partial S}{\partial y} = x - \dfrac{64}{y^2} = 0.$

Solving these, we get $x = 4$; $y = 4$.

Also $l = \dfrac{\partial^2 S}{\partial x^2} = \dfrac{128}{x^3}$, $m = \dfrac{\partial^2 S}{\partial x \partial y} = 1$; $n = \dfrac{\partial^2 S}{\partial y^2} = \dfrac{128}{y^2}$

At $x = 4$ & $y = 4$, $ln - m^2 = \dfrac{128}{x^3} \times \dfrac{128}{y^3} - 1 = 2 \times 2 - 1 = 3 > 0$ and $l = \dfrac{128}{x^3} = 2 > 0$

Thus, S is minimum when $x = 4, y = 4$.

From (2), we get $z = 2$

$\therefore$ The dimensions of the box for least material for its construction are 4, 4, 2.

Example : Find the points on the surface $z^2 = xy + 1$ that are nearest to the origin.

Solution : Let P (x, y, z) be any point on the surface

$$\phi(x,y,z) = z^2 - xy - 1 = 0 \qquad \dots (1)$$

Let $OP = p = \sqrt{x^2 + y^2 + z^2} \qquad \dots (2)$

We have to find the minimum values of (2) subject to the condition (1).

From (1) and (2), we have

$$p^2 = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1 \qquad \dots (3)$$

Let $r = \frac{\partial^2 p}{\partial x^2}, s = \frac{\partial^2 p}{\partial x \, \partial y}$ and $t = \frac{\partial^2 p}{\partial y^2}$

Differentiating (3) partially w.r.t 'x' and 'y', we get

$$2p\frac{\partial p}{\partial x} = 2x + y \qquad \dots (4)$$

and $$2p\frac{\partial p}{\partial y} = 2y + x \qquad \dots (5)$$

The critical points are given by $\frac{\partial p}{\partial x} = 0$ and $\frac{\partial p}{\partial y} = 0$

$\Rightarrow \quad 2x + y = 0$ and $2y + x = 0 \Rightarrow x = 0, y = 0$

$(1) \Rightarrow \quad z = \pm\sqrt{xy + 1} = \pm 1 \quad (\because x = 0, y = 0)$

$\therefore$ P (0, 0, 1) and Q (0, 0, −1) are the critical points of p.

Differentiating (4) partially w.r.t. 'x' and 'y', we get

$$2pr + 2\left(\frac{\partial p}{\partial x}\right)^2 = 2 \Rightarrow r = \frac{2}{2p} = 1 \text{ at } P \text{ and } Q \quad \left(\because p = 1 \text{ and } \frac{\partial p}{\partial x} = 0 \text{ at } P \text{ and } Q\right)$$

and $$2ps + 2\frac{\partial p}{\partial x}\cdot\frac{\partial p}{\partial y} = 1 \Rightarrow s = \frac{1}{2p} = \frac{1}{2} \text{ at } P \text{ and } Q \quad \left(\because p = 1, \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \text{ at } P \text{ and } Q\right)$$

Diff. (5) partially w.r.t. 'y', we get

$$2pt + 2\left(\frac{\partial p}{\partial y}\right)^2 = 2 \Rightarrow t = \frac{2}{2p} = 1 \text{ at P and Q.}$$

$\therefore$ At P and Q, $rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$

Hence p has minimum at P and Q.

$\therefore$ Required points are (0, 0, 1) and (0, 0, −1).