Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x - h)^2 + k$

Student Outcomes

- Students graph simple quadratic equations of the form $y = a(x - h)^2 + k$ (completed-square or vertex form), recognizing that (h, k) represents the vertex of the graph and use a graph to construct a quadratic equation in vertex form.
- Students understand the relationship between the leading coefficient of a quadratic function and its concavity and slope and recognize that an infinite number of quadratic functions share the same vertex.

MP.7 & MP.8

The Opening Exercise is set up so students get three different quadratic equations and they are asked to analyze the similarities and differences in the structure of their equations and the corresponding graphs, which leads to a generalization about the graphs of quadratic equations of the form $y = (x - h)^2$.

Scaffolding:

- Recall: What does the graph of a quadratic equation look like?
- Remind students that when calculating squares in their calculator they need to watch out for a common error: $(-1)^2 \neq -1^2$.

Lesson Notes

Throughout this lesson we use "$y =$" notation when we are talking about the equation that represents the function and "$f(x) =$ " when talking about the function itself. It may be important to review the reason for using both notations and the difference between them. Students will need a graphing calculator and graph paper for this lesson.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

Graph the equations $y = x^2$, $y = (x - 2)^2$, and $y = (x + 2)^2$ on the interval $-3 \leq x \leq 3$.

Have students graph the equations, $y = x^2$, $y = (x - 2)^2$, and $y = (x + 2)^2$ on the interval $-3 \leq x \leq 3$. You might have them graph using the graphing calculator or graph paper.

- How are the graphs of these equations similar? How are they different?
  - *The graphs look similar in that they have the same shape. Point out that the graphs are all translations of each other. They have different vertices: $y = x^2$ has its vertex at $(0,0)$, while $y = (x - 2)^2$ has its vertex at $(2,0)$ and $y = (x + 2)^2$ has its vertex at $(-2, 0)$.*

Scaffolding:

- For students who quickly grasp the horizontal shift implied in the Opening Exercise: We already know how to move the vertex left and right. How might we move it up and down?
- The function, $f(x) = x^2$, is called the "parent function" for all quadratic functions and their graphs.

- Now consider the graph of $y = (x-5)^2$. Where would you expect this graph to be in relation to the other two?
  - *The graph of* $y = (x-5)^2$ *is* 5 *units to the right of the graph of* $y = x^2$ *and* 3 *units to the right of* $y = (x-2)^2$.

Discussion (10 minutes)

Based on the Opening Exercise and the lessons in Module 3 (e.g., see horizontal translations in Module 3, Lesson 18), students should be able to reason that replacing x with $(x+N)$ moves the vertex N units to the right (for a negative N) or the left (for a positive N) on the x-axis. Push further. Allow students to discuss the following questions with their partner or small group before taking suggested answers from the class:

- Why do you think the graph moves to the *right* when we *subtract* a positive number from x inside the parentheses and to the *left* when we *add*?
  - *As an example, let's start with a quadratic with vertex location* $(0, k)$*, giving us the equation:* $y = x^2 + k$*. After a horizontal translation,* $x \rightarrow (x-h)$*, the height of the vertex should remain the same, namely* $y = k$*. That means* $y = (x-h)^2 + k = k$ *at the vertex. We are curious about where the new vertex is horizontally; that is, what* x*-value will make the previous equation true. This implies* $(x-h)^2 = 0$*; therefore,* $x = h$*. That means, after the translation* $x \rightarrow (x-h)$*, the vertex* $(0, k)$ *is translated to* (h, k)*, and the whole graph is translated* h *units to the right. Since* h *is positive, the graph shifts to the right; if* h *was negative,* $(x-h)$ *will read as* x *plus a positive number and the graph shifts to the left.*
- Your teacher is 6 units tall and standing at the position $x = -2$ on a horizontal axis. Is it possible to find a quadratic equation that looks just like $y = x^2$, but that sits directly on top of your teacher's head?

  With a partner, take 5 minutes and experiment to see if you can find the quadratic equation to represent this situation. Use what we have already learned in earlier lessons and modules to help you get started. Construct tables and draw graphs to verify your results. Remind students that they performed vertical translations in Module 3, Lesson 17.
  - *Students should be able to discover that, in addition to moving the graph left or right by adding or subtracting within the parentheses (adjusting the horizontal position), they can move up and down by adding or subtracting outside the parentheses (adjusting the vertical position).*

Scaffolding:

- Visual learners will benefit from experimenting with their graphing calculators to determine the effect of changing the values of h and k in equations of the form, $y = a(x+h)^2 + k$.
- For now let the leading coefficient $= 1$.

MP.4 In the activity above, students model the situation using tables and graphs. Then they conclude that the graphs of the equations can move up or down by adding or subtracting a constant outside the parentheses.

Exercises 1–2 (8 minutes)

Exercises 1–2

1. **Without graphing, state the vertex for each of the following quadratic equations.**

   a. $y = (x - 5)^2 + 3$

      $(5, 3)$

   b. $y = x^2 - 2.5$

      $(0, -2.5)$

   c. $y = (x + 4)^2$

      $(-4, 0)$

2. **Write a quadratic equation whose graph will have the given vertex.**

   a. $(1.9, -4)$

      $y = (x - 1.9)^2 - 4$

   b. $(0, 100)$

      $y = x^2 + 100$

   c. $\left(-2, \frac{3}{2}\right)$

      $y = (x + 2)^2 + \frac{3}{2}$

Scaffolding:

If students need more examples to reinforce this concept, have them compare the following graphs either on their graphing calculator or graph paper:

$$y = \frac{1}{2}(x - 1)^2$$
$$y = 2(x - 1)^2$$
$$y = -2(x - 1)^2$$

Ask them to comment on how the three graphs are similar and how they are different. Hopefully, they will notice that all three have the same vertex but some are stretched or shrunk vertically, and some open down rather than up.

Discussion (6 minutes)

Review the problems above, and when discussing solutions for Exercise 2 ask the following:

- Are these the *only* quadratic equations with graphs with these vertices? Is there another way to write two equations that have the same vertex but are different?

Using the equation from Exercise 2(b), ask students to experiment with ways to change the graph without changing the vertex. Encourage them to write equations, evaluate them using a table, and graph the results.

Students come to the conclusion that if we multiply the term $(x - h)^2$ by some other number, we keep the vertex the same but the graph experiences a vertical stretch or shrink, or in the case of a negative coefficient, the direction that the graph opens is reversed.

Verify that this is true by applying the same rule to the equation from Exercise 2(c).

- Can we generalize and discuss the effect the leading coefficient, *a*, has on the graph of $f(x) = a(x - h)^2 + k$? Compared to the graph when $a = 1$:
  - *The graph is shrunk vertically when* $-1 < a < 1$.
  - *The graph is stretched vertically when* $a < -1$ *or* $a > 1$.
  - *The graph opens up when* a *is positive.*
  - *The graph opens down when* a *is negative.*

Example 1 (8 minutes)

This application problem about rectangular area connects vertex form to a real-world context. You may consider asking students to create the graph of the function in part (b) to connect problem's solution explicitly to the graph of a quadratic function. To aid in comprehension of the problem, consider asking students to name some hypothetical lengths and widths of the pen given the 60-foot constraint (e.g., 20 by 10, 5 by 25, etc.).

Example 1

Caitlin has 60 feet of material that can be used to make a fence. Using this material, she wants to create a rectangular pen for her dogs to play in. What dimensions will maximize the area of the pen?

a. Let w be the width of the rectangular pen in feet. Write an expression that represents the length when the width is w feet.

$\frac{(60-2w)}{2}$ or $30-w$

b. Define a function $A(w)$ that describes the area, A, in terms of the width, w.

$A(w) = w(30-w)$ or $A(w) = -w^2 + 30w$

c. Rewrite $A(w)$ in vertex form.

$$A(w) = -w^2 + 30w$$
$$= -(w^2 - 30w)$$
$$= -(w^2 - 30w + 225) + 225$$
$$= -(w-15)^2 + 225$$

d. What are the coordinates of the vertex? Interpret the vertex in terms of the problem.

The vertex is located at $(15, 225)$. Since the leading coefficient is negative, the function has a maximum. The maximum value of the function is 225, which occurs when $w = 15$. For this problem, this means that the maximum area is 225 square feet, which happens when the width is 15 feet.

e. What dimensions maximize the area of the pen? Do you think this is a surprising answer?

The pen has the greatest area when the length and width are both 15 feet. Students may or may not be surprised to note that this occurs when the rectangle is a 15×15 square.

Closing (3 minutes)

- How many quadratic equations are there whose graphs share a given vertex?
  - *Infinitely many because there are an infinite number of values for the value of the leading coefficient, a.*

Lesson Summary

When graphing a quadratic equation in vertex form, $y = a(x-h)^2 + k$, (h, k) are the coordinates of the vertex.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x - h)^2 + k$

Exit Ticket

1. Compare the graphs of the function, $f(x) = -2(x + 3)^2 + 2$ and $g(x) = 5(x + 3)^2 + 2$. What do the graphs have in common? How are they different?

2. Write two different equations representing quadratic functions whose graphs have vertices at $(4.5, -8)$.

Exit Ticket Sample Solutions

1. **Compare the graphs of the function, $f(x) = -2(x+3)^2 + 2$ and $g(x) = 5(x+3)^2 + 2$. What do the graphs have in common? How are they different?**

   Both quadratic equations have their vertices at $(-3, 2)$. However, the graph of $f(x)$ has less vertical stretch that the graph of $g(x)$, and the graph of $f(x)$ opens down, whereas the graph of $g(x)$ opens up.

2. **Write two different quadratic equations whose graphs have vertices at $(4.5, -8)$.**

   $y = (x-4.5)^2 + 8$ and $y = -(x-4.5)^2 + 8$ or any similar responses with different leading coefficients.

Problem Set Sample Solutions

1. **Find the vertex of the graphs of the following quadratic equations.**
   - a. $y = 2(x-5)^2 + 3.5$

     $(5, 3.5)$
   - b. $y = -(x+1)^2 - 8$

     $(-1, -8)$

2. **Write a quadratic equation to represent a function with the following vertex. Use a leading coefficient other than 1.**
   - a. $(100, 200)$

     $y = -2(x-100)^2 + 200$
   - b. $\left(-\frac{3}{4}, -6\right)$

     $y = 4\left(x + \frac{3}{4}\right)^2 - 6$

3. **Use vocabulary from this lesson (i.e., stretch, shrink, opens up, opens down, etc.) to compare and contrast the graphs of the quadratic equations $y = x^2 + 1$ and $y = -2x^2 + 1$.**

   The quadratic equations share a vertex at $(0, 1)$, but the graph for the equation $y = -2x^2 + 1$ opens down and has a vertical stretch, while the graph of the equation $y = x^2 + 1$ opens up.